Math 21b: Calculus
Third Midterm Solution

(Woei 2/27/04)

Write your name and student ID number in the upper right hand corner of this sheet, or the first page of your answers. Initial your other pages.

This is a closed book, no calculator test.

1. (40 points) Do these indefinite integrals:

a. $\int \frac{1}{x^2+x+2}dx$

Solution: Completing the square for x^2+x+2, we get:

$$x^2+x+2 = \left(x+\frac{1}{2}\right)^2 + \frac{7}{4}$$

Therefore,

$$\int \frac{dx}{x^2+x+2} = \int \frac{dx}{(x+\frac{1}{2})^2+\frac{7}{4}} = \int \frac{dx}{\frac{7}{4}\left(\frac{4}{7}(x+\frac{1}{2})^2+1\right)} = \frac{4}{7}\int \frac{dx}{\left(\frac{2}{\sqrt{7}}(x+\frac{1}{2})\right)^2+1}$$

Let

$$u = \frac{2}{\sqrt{7}}\left(x+\frac{1}{2}\right) \Rightarrow du = \frac{2}{\sqrt{7}}dx \Rightarrow \frac{\sqrt{7}}{2}du = dx$$

Thus

$$\frac{\sqrt{7}}{2}\frac{4}{7}\int \frac{du}{u^2+1} = \frac{2\sqrt{7}}{7}\arctan u + C = \frac{2\sqrt{7}}{7}\arctan\frac{2(x+\frac{1}{2})}{\sqrt{7}} + C$$

b. $\int (\cos 2x)e^x dx$

Solution: Using integration by parts twice, we get a recursion:

Solution 1:

$$\int \cos(2x)e^x dx \quad \Rightarrow \quad \begin{matrix} u = e^x & dv = \cos(2x)dx \\ du = e^x dx & v = \frac{\sin(2x)}{2} \end{matrix}$$

$$\int \cos(2x)e^x dx = e^x\frac{\sin(2x)}{2} - \int \frac{sin(2x)}{2}e^x dx \quad \Rightarrow \quad \begin{matrix} u = \frac{e^x}{2} & dv = \sin(2x)dx \\ du = \frac{e^x}{2}dx & v = -\frac{\cos(2x)}{2} \end{matrix}$$

$$\int \cos(2x)e^x dx = e^x\frac{\sin(2x)}{2} - \left[-\frac{e^x}{2}\frac{\cos(2x)}{2} - \int -\frac{e^x}{2}\frac{cos(2x)}{2}dx\right] \Longrightarrow$$

$\int \cos(2x)e^x dx = \frac{e^x \sin(2x)}{2} + \frac{e^x \cos(2x)}{4} - \frac{1}{4}\int e^x \cos(2x)dx \Longrightarrow$
$\left(1+\frac{1}{4}\right)\int e^x \cos(2x)dx = \frac{e^x \sin(2x)}{2} + \frac{e^x \cos(2x)}{4} + C \Longrightarrow$

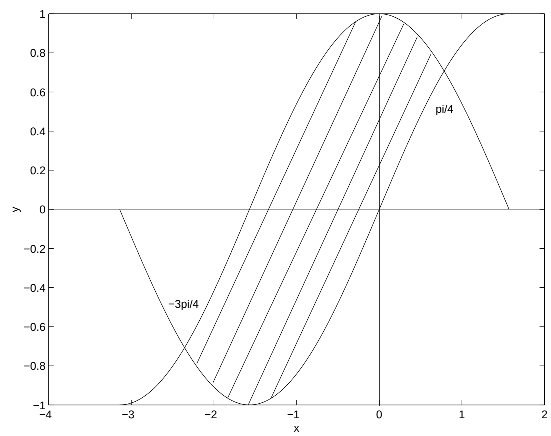
Finally,

$$\int \cos(2x)e^x dx = \frac{2}{5}e^x \sin(2x) + \frac{1}{5}e^x \cos(2x) + C$$

2. (20 points) Draw the set of points (x,y) in the plane with $-\frac{3\pi}{4} \leq x \leq \frac{\pi}{4}$ and $\sin x \leq y \leq \cos x$. Then find its area.

Solution:

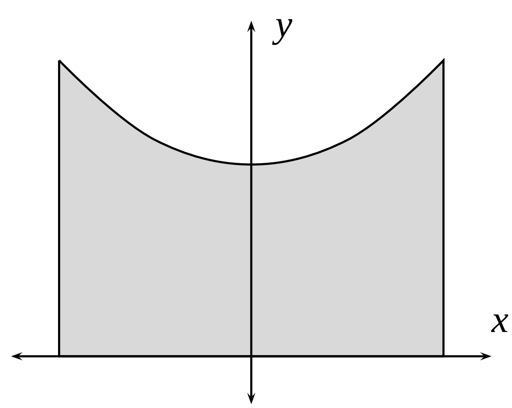
$$\int_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} \cos(x) - \sin(x)dx = sin(x) + cos(x)\Big|_{-\frac{3\pi}{4}}^{\frac{\pi}{4}} = \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$



3. (30 points) Recall the function

$$\cosh x = \frac{e^x + e^{-x}}{2}.$$

Let R be the region of points in the plane with $0 \leq y \leq \cosh x$ and $-1 \leq x \leq 1$. It looks like this:

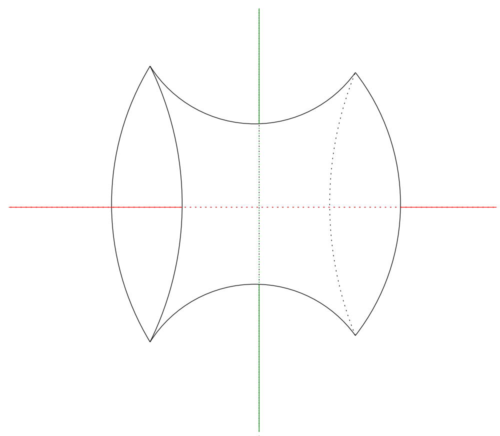


Revolve R around the x axis in 3 dimensions to make a solid of revolution. Draw it and find its volume.

Solution: Using Concentric Circle Technique, i.e. $\pi r^2 dx$

$$\pi\int_{-1}^{1}\cosh^2(x)dx = \pi\int_{-1}^{1}\left(\frac{e^x+e^{-x}}{2}\right)^2 dx = \frac{\pi}{4}\int_{-1}^{1} e^{2x}+e^{-2x}+2dx =$$

$$\frac{\pi}{4}\left[2\int_0^1 e^{2x}+e^{-2x}+2dx\right] = \frac{\pi}{2}\left[\frac{e^{2x}}{2}-\frac{e^{-2x}}{2}+2x\Big|_0^1\right] = \frac{\pi}{2}\left[\frac{e^2}{2}-\frac{e^{-2}}{2}+2\right] = \frac{\pi}{4}\left[e^2-e^{-2}\right]+\pi$$



4. (30 points) Revolve the region R from problem 3 around the y axis. Draw this solid of revolution and find its volume.

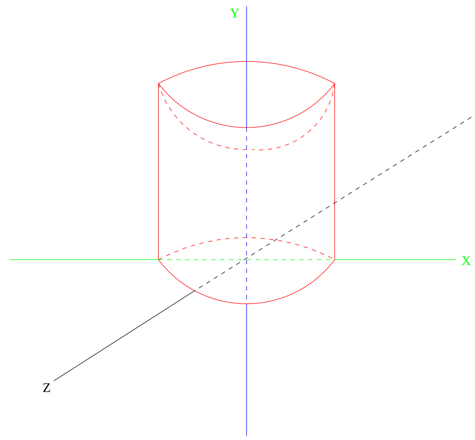
Solution: Using Shell Technique, i.e. $2\ \pi x r dx$

$$2\pi\int_0^1 x\cosh(x)dx = \pi\int_0^1 x(e^x+e^{-x})dx \qquad \Rightarrow \qquad \begin{array}{ll} u = x & dv = e^x+e^{-x} \\ du = dx & v = e^x - e^{-x} \end{array}$$

$$\pi\int_0^1 x(e^x+e^{-x})dx = \pi\left[x(e^x-e^{-x})\Big|_0^1 - \int_0^1 e^x - e^{-x}dx\right] =$$

$$\pi\left[e-e^{-1}-\left[e^x+e^{-x}\big|_0^1\right]\right] = \pi\left[e-e^{-1}-\left[e+e^{-1}-2\right]\right] =$$

$$\pi\left[2-2e^{-1}\right]$$



Y
X
Z