Math 21b: Calculus
First Midterm Solution
(Woei 1/28/04)

This is a closed book, no calculator test. Remember, if a question has English in it, so should the answer.

1. (20 points) Write the sum $50+51+52+\ldots+150$ in Σ notation and also compute the answer.

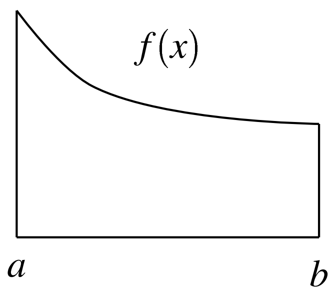
Solution: $\sum_{i=50}^{150} i$

Using Gauss's trick, we get:

$$\begin{array}{r} 50+\ 51+\ 52+\ldots+150 \\ 150+149+148+\ldots+\ 50 \\ \hline 200+200+200+\ldots+200 \end{array}$$

There are 101 *(150-50+1)* terms, thus the sum is $\frac{101*200}{2}$ = 10,100

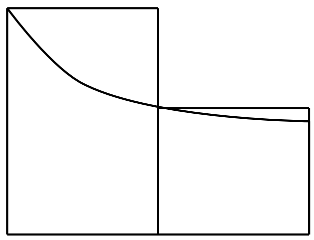
2. (20 points) A function $f(x)$ looks like this:



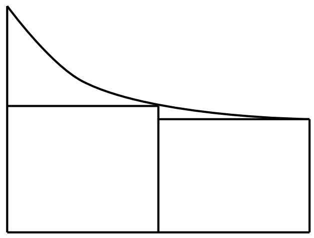
Explain which of the following rules for approximating $\int_a^b f(x)dx$ are underestimates and which are overestimates: the left endpoint rule, the right endpoint rule, the midpoint rule, the trapezoid rule.

Solution: The function $f(x)$ is decreasing and $f(x)$ is concave up, we can conclude:

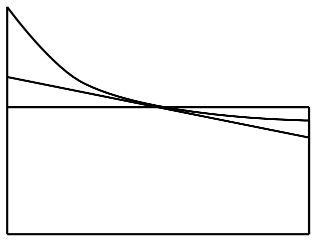
The left endpoint rule is an overestimate:



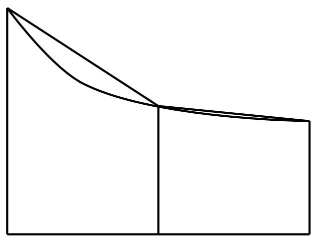
The right endpoint rule is an underestimate:



Since $f(x)$ is concave up, then any tangent line to curve will be under the curve, therefore the midpoint rule will give an underestimate:



The function $f(x)$ is decreasing and concave up, thus the trapezoid rule gives an overestimate:



3. (30 points) Do these definite integrals:

a. $\int_{-1}^{1}(1+x)^2(1-x)^2dx$

Solution: Note $f(x)=(1+x)^2(1-x^2)$ is even, since $f(-x)=(1-x)^2(1+x)^2=f(x)$.

$$\begin{aligned}\int_{-1}^{1}(1+x)^2(1-x)^2dx &= 2\int_0^1(1+x)^2(1-x)^2dx\\ &= 2\int_0^1((1+x)(1-x))^2dx\\ &= 2\int_0^1(1-x^2)^2\\ &= 2\int_0^1 1-2x^2+x^4dx\\ &= 2\left[x-\frac{2}{3}x^3+\frac{x^5}{5}\Big|_0^1\right]\\ &= \frac{16}{15}\end{aligned}$$

b. $\int_{-1}^{1}\frac{x}{x^4+1}dx$

Solution: Note $f(x)=\frac{x}{x^4+1}$ is odd, since $f(-x)=\frac{-x}{x^4+1}=-f(x)$, thus $\int_{-1}^{1}\frac{x}{x^4+1}dx=0$. Using substitution, let $u=x^2\Rightarrow du=2xdx\Rightarrow\frac{1}{2}\int_1^1\frac{du}{u^2+1}=0$. Since x goes from $-1\to 1$ then u goes from $1\to 1$. If you decided to go even further and carry out the integration, you would get: $\frac{1}{2}\int_1^1\frac{du}{u^2+1}=\arctan(u)\big|_1^1=0$. And if you decided to substitute back, you would get: $\frac{1}{2}\int_1^1\frac{du}{u^2+1}=\arctan(u)\big|_1^1=\arctan(x^2)\big|_{-1}^1=0$

c. $\int_0^{\pi/2}\frac{\cos 3x}{2}dx$

Solution:

$$\int_0^{\frac{\pi}{2}}\frac{\cos 3x}{2}dx=\frac{\sin 3x}{6}\Big|_0^{\frac{\pi}{2}}=-\frac{1}{6}$$

Let $u=3x\Rightarrow du=3xdx\Rightarrow\int_0^{\pi/2}\frac{\cos 3x}{2}dx=\frac{1}{6}\int_0^{\frac{3\pi}{2}}\cos u du=\frac{1}{6}\sin u\Big|_0^{\frac{3\pi}{2}}=-\frac{1}{6}$

4. (30 points) Do these indefinite integrals:

a. $\int(1+e^x)(1-e^{-x})dx$

Solution:

$$\int(1+e^x)(1-e^{-x})dx=\int 1-e^{-x}+e^x-e^xe^{-x}dx=\int 1-e^{-x}+e^x-e^{x-x}dx=$$

$$\int 1-e^{-x}+e^x-1dx=\int e^x-e^{-x}dx=e^x+e^{-x}+C$$

b. $\int(\cos x)(\tan x)dx$

Solution:

$$\int(\cos x)(\tan x)dx=\int(\cos x)(\frac{\sin x}{cosx})dx=\int\sin xdx=-\cos x+C$$

c. $\int e^xe^{e^x}dx$

Solution: Let

$$u=e^x\Rightarrow du=e^xdx\Rightarrow\int e^{e^x}e^xdx=\int e^udu=e^u+C=e^{e^x}+C$$

5. (20 points) There is a super-Simpson's rule which, in one convenient case, says that:

$$\int_{-1}^{1} f(x)dx \approx \frac{1}{6}f(1) + \frac{5}{6}f(\frac{1}{\sqrt{5}}) + \frac{5}{6}f(-\frac{1}{\sqrt{5}}) + \frac{1}{6}f(-1).$$

Take it as a given that

$$\int_{-1}^{1} \frac{2}{1+x^2}dx = \pi.$$

Use this super-Simpson's rule to find an approximation for π. (You can leave it as a mixed fraction.)

Solution: Just plug it in! $f(x) = \frac{2}{1+x^2}$. Using super simpson's method we get

$$\int_{-1}^{1} \frac{2}{1+x^2}dx \approx \frac{1}{6} + \frac{5}{6}\frac{2}{1+\frac{1}{5}} + \frac{5}{6}\frac{2}{1+\frac{1}{5}} + \frac{1}{6} = \frac{1}{3} + \frac{25}{9} = \frac{28}{9} = 3\frac{1}{9} \approx \pi$$