Instr.: *Woei* April 27, 2006

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No Calculators.

The series

$$\frac{2}{x-1} = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$$

converges to $\frac{2}{x-1}$ for $1 < x < 5$.

1. (*8 pts*) What series do you get if you integrate the series term by term? What is it sum? For what values of x does the new series converge?

Let $f(x) = \frac{2}{x-1} = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$

Then $\int f(x)\,dx = \int \frac{2}{x-1}\,dx = 2\ln|x-1| + C_L =$

$$= x - \frac{1}{2}\frac{(x-3)^2}{2} + \frac{1}{4}\frac{(x-3)^3}{3} + \cdots + \left(-\frac{1}{2}\right)^n \frac{(x-3)^{n+1}}{n+1} + \cdots + C_R$$

For $x = 3$, then $2\ln 2 + C_L = 3 + C_R$ Let $C = C_R - C_L$

Then $2\ln|x-1| - 2\ln 2 = x - 3 - \frac{1}{2}\frac{(x-3)^2}{2} + \frac{1}{4}\frac{(x-3)^3}{4} + \cdots + \left(-\frac{1}{2}\right)\frac{(x-3)^{n+1}}{n+1} + \cdots$

$$= \sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n \frac{(x-3)^{n+1}}{n+1}$$

From Term by Term Integration Theorem $\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \frac{(x-3)^{n+1}}{n+1}$ converges for $1 < x < 5$. For $x = 1$ $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \frac{(-2)^{n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-2}{n+1}$ which diverges. For $x = 5$ $\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \frac{2^{n+1}}{n+1} = 2\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ which converges by Alternating Series Test. Therefore the interval of convergence is $1 < x \le 5$.

2. (*2 pts*) What series do you get if you differentiate the series term by term?

Let $f(x) = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 + \cdots + \left(-\frac{1}{2}\right)^n (x-3)^n + \cdots$

Then $f'(x) = -\frac{1}{2} + \frac{2}{4}(x-3) + \cdots + \left(-\frac{1}{2}\right)^n n(x-3)^{n-1} + \cdots$

$$= \sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n n(x-3)^{n-1}$$