Quiz 4 | 6:00 – 7:00pm

The series

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

converges to $\sin x$ for all x.

a) Find ~~this~~ find the series for $\cos x$. For what values of x should the series converge?

b) By replacing x by $2x$ in the series for $\sin x$, find a series that converges to $\sin 2x$ for all x.

Solution

a) Take the derivative of $\sin x$

$$\cos x = \frac{d}{dx}\sin x = \frac{d}{dx}x + \frac{d}{dx}\left(-\frac{x^3}{3!}\right) + \frac{d}{dx}\left(\frac{x^5}{5!}\right) + \frac{d}{dx}\left(-\frac{x^7}{7!}\right) + \dots + (-1)^n\frac{d}{dx}\left(\frac{x^{2n+1}}{(2n+1)!}\right) + \dots$$

$$= 1 - \frac{3x^2}{3!} + \frac{5}{5!}x^4 - \frac{7x^6}{7!} + \dots + (-1)^n\frac{2n+1}{(2n+1)!}x^{2n} + \dots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^n\frac{x^{2n}}{(2n)!} + \dots$$

The series converges for all values of x since

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{2n!}{x^{2n}}\right| = x^2\lim_{n\to\infty}\frac{1}{(2n+1)(2n+2)} = 0 < 1, \text{ for all } x.$$

b)
$$\sin(2x) = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots + \frac{(-1)^n(2x)^{2n+1}}{(2n+1)!} + \dots$$

$$= 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \frac{128}{7!}x^7 + \dots + \frac{(-1)^n(2)^{2n+1}}{(2n+1)!}x^{2n+1} + \dots$$