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1. (*5 pts*) Find a formula for the nth partial sum of

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{(n+1)(n+2)}+\cdots$$

and use it to find the series sum, if the series converges.

The nth partial sum is $S_n = \sum_{i=1}^{n} \frac{1}{(i+1)(i+2)}$. Writing $\frac{1}{(i+1)(i+2)}$ as $\frac{A}{i+1} + \frac{B}{i+2}$, i.e. $\frac{1}{(i+1)(i+2)} = \frac{A}{i+1} + \frac{B}{i+2} \Rightarrow 1 = A(i+2) + B(i+1)$

$= (A+B)i + 2A + B$

$\Rightarrow \begin{array}{l} A+B=0 \Rightarrow A=-B \\ 2A+B=1 \Rightarrow -2B+B=1 \Rightarrow B=-1 \Rightarrow A=1 \end{array}$, so $S_n = \sum_{i=1}^{n} \frac{1}{i+1} + \frac{-1}{i+2}$

$= \frac{1}{2} + \frac{-1}{3} + \frac{1}{3} + \frac{-1}{4} + \cdots + \frac{1}{n+1} + \frac{-1}{n+2} = \frac{1}{2} + \frac{-1}{n+2}$

Therefore as $n \to \infty$, $S_n \to \frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ which converges

2. (*5 pts*) Does

$$\sum_{n=1}^{\infty} \frac{2}{1+e^n}$$

converge or diverge? Give reasons for your answer.
Hint: Substitution of $u = e^n$ and partial fractions.

To check if $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ converges / diverges we use integral test

i.e. we look at $\int_1^{\infty} \frac{2}{1+e^n}\, dn$ if this integral is finite

then $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ converges if not it diverges.

$\int_1^{\infty} \frac{2}{1+e^n}\, dn \overset{\substack{u=e^n \\ du = e^n dn}}{=} \int_{e}^{\infty} \frac{2}{1+u} \frac{du}{u}$ using partial fractions $\frac{2}{u(1+u)} = \frac{A}{u} + \frac{B}{1+u}$

$\Rightarrow 2 = A(1+u) + Bu$

$= (A+B)u + A$

$\Rightarrow A = 2 \quad A+B=0 \Rightarrow B=-2$

so $\int_{e}^{\infty} \frac{2}{u(1+u)}\, du = \int_{e}^{\infty} \frac{2}{u} + \frac{-2}{1+u}\, du$

$= 2\ln u - 2\ln(1+u) \Big|_e^{\infty} = 2\ln\left(\frac{u}{1+u}\right)\Big|_e^{\infty} = 2\ln 1 - 2\ln\frac{e}{1+e} = 2\left(\ln(1+e) - 1\right) > 0$

Therefore $\sum_{n=1}^{\infty} \frac{2}{1+e^n}$ $\boxed{\text{converges}}$!