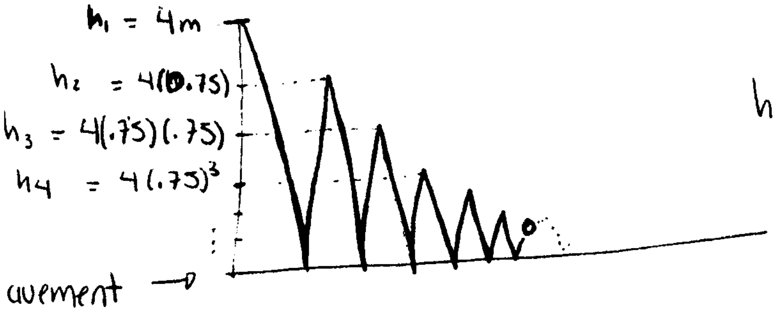
Quiz 2

1: A ball is dropped from a height of 4 m. Eeach time it strikes the pavement after falling from a height of h meters it rebounds to a height of 0.75 h meters. Find the total distance the ball travels up and down.

Solution: The ball describes the following movement:



$h_n = 4(.75)^n \qquad n = 1, 2, 3 \ldots$

Let D = total distance the ball travels <u>up</u> and <u>down</u>

Then $D = h_1 + h_2 + h_2 + h_3 + h_3 + h_4 + h_4 + \cdots h_n + h_n + \cdots$

$$= h_1 + 2(h_2 + h_3 + h_4 + \cdots + h_n + \cdots)$$

$$= 4 + 2\left(4(.75) + 4(.75)^2 + 4(.75)^3 + \cdots + 4(.75)^n + \cdots\right)$$

$$= 4 + 2\Bigg[\underbrace{\sum_{n=1}^{\infty} 4(.75)^n}_{\text{almost geometric series since we are missing the term } a_0 = 4 \text{, but we can fix it.}}\Bigg], \quad a = 4 \quad r = .75 < 1$$

$$= 4 + 2\left[\sum_{n=0}^{\infty} 4(.75)^n - 4\right]$$

$$= 4 + 2\left[\frac{4}{1 - .75} - 4\right] = 4 + 2\left[\frac{4}{.25} - 4\right] = 4 + 2\left(\frac{4}{1/4} - 4\right]$$

$$= 4 + 2(16 - 4) = \boxed{28 \text{ m}}$$

(2) Converges or diverges? Give reason for your answer

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

Solution

We will use integral test: Let $a_n = \frac{n}{n^2+1}$ and $f(x) = \frac{\cancel{x}}{x^2+1}$

- f is continuous for every real number
- f is positive for every number bigger than zero
- $f(n) = a_n$
- f is decreasing. Let us check it using first derivative.

(recall f decreasing if $f' < 0$).

$$f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

For $x>1$ we have $x^2>1$
$-x^2+1 < 0$

therefore $f'(x) < 0 \quad \forall x>1$

Since all hypothesis of the Integral test hold, we just need to see if $\int_1^{\infty} f(x)\,dx$ converges or diverges

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_2^{\infty} \frac{du}{u} = \frac{1}{2}\lim_{b\to\infty}\int_2^b \frac{du}{u} = \frac{1}{2}\lim_{b\to\infty} \ln u\Big|_2^b =$$

$u = x^2+1$
$du = 2x\,dx$
if $x=1$ $u=2$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b) - \ln(2)\right]$$

$$= \infty$$

Therefore $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges