Answers (6:10pm – 7:00 pm)

Decide the convergence or divergence of the following two sequences.

1) $a_n = \dfrac{2^n - 1}{3^n}$

First of all notice that we can rewrite a_n as follows

$$a_n = \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n.$$

By the difference rule we have that

$$\lim_{n\to\infty} \left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n - \lim_{n\to\infty} \left(\frac{1}{3}\right)^n$$

(We can do this since the limits exist as we will see below)

And we know that $\lim_{n\to\infty} x^n = 0$ if $|x| < 1$. Thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n - \lim_{n\to\infty} \left(\frac{1}{3}\right)^n = 0 + 0 = 0$$

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2) $a_n = \dfrac{(\sin 2n)^2}{2^n}$

First, observe that $0 \le \sin^2 2n \le 1$ for all $n \in \mathbb{N}$. This implies that

$$0 = \frac{0}{2^n} \le \frac{\sin^2 2n}{2^n} \le \frac{1}{2^n}$$

Since $\lim_{n\to\infty} 0$ and $\lim_{n\to\infty} \frac{1}{2^n}$ exist, using the sandwich rule we have

$$0 = \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\sin^2 2n}{2^n} \le \lim_{n\to\infty} \frac{1}{2^n} = 0$$

Therefore $\lim_{n\to\infty} \dfrac{\sin^2 2n}{2^n} = 0$.