(5:10pm - 6:00pm)

Answers

Decide the convergence or divergence of the following two sequences. Find the limit of each convergent sequence.

1) $a_n = \left(2 - \frac{1}{2^n}\right)\left(3 + \frac{1}{2^n}\right)$.

First notice that we can rewrite a_n in the following way

$$a_n = \left(2 - \left(\tfrac{1}{2}\right)^n\right)\left(3 + \left(\tfrac{1}{2}\right)^n\right)$$

Using by the product rule we have that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 - \left(\tfrac{1}{2}\right)^n\right) \cdot \lim_{n\to\infty}\left(3 + \left(\tfrac{1}{2}\right)^n\right)$$

(We can do this since the limit of each factor exists as we will see below).

Now observe that we can also use the difference and sum rules for each limit respectively. So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(2 - \left(\tfrac{1}{2}\right)^n\right) \cdot \lim_{n\to\infty}\left(3 + \left(\tfrac{1}{2}\right)^n\right) = \left(2 - \lim\left(\tfrac{1}{2}\right)^n\right) \cdot \left(3 + \lim\left(\tfrac{1}{2}\right)^n\right)$$

$$= (2-0)(3+0) = 2 \cdot 3 = 6$$

Since $\lim_{n\to\infty} x^n = 0$ if $|x|<1$.

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2) $a_n = \frac{\cos^2 n}{2^n}$

First notice that $0 \le \cos^2 n \le 1$ for all $n \in \mathbb{N}$. This implies that

$$0 = \frac{0}{2^n} \le \frac{\cos^2 n}{2^n} \le \frac{1}{2^n} \text{ for all } n$$

By the sandwich rule (since $\lim_{n\to\infty} 0$ and $\lim_{n\to\infty} \frac{1}{2^n}$ exist), we have

$$0 = \lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{\cos^2 n}{2^n} \le \lim_{n\to\infty} 2^n = 0$$

Thus $\lim_{n\to\infty} \frac{\cos^2 n}{2^n} = 0$.