Solutions to Practice Final
Math 21A

Ernest Woei

December 8, 2003

Problem 1 Use differentials to approximate $\log_2(\ln(e+1))$.
Solution:
Recall: $f(x+h) \approx f(x) + h * f'(x)$ *for small h.*
First we will approximate

$$\ln(e+1)$$

Let $f(x) = \ln(x)$, then $f'(x) = \frac{1}{x}$.

Choose: $x = e$ and $h = 1$ since differentials work *for small h* and $1 < e$. Therefore

$$f(e+1) \approx f(e) + 1 * f'(e) = \ln(e) + 1 * \frac{1}{e} = 1 + \frac{1}{e}$$

We will now approximate

$$\log_2(1 + \frac{1}{e})$$

Let $f(x) = \log_2(x)$, then $f'(x) = \frac{1}{\ln(2)*x}$.

Choose $x = 1$ and $h = \frac{1}{e}$, then

$$f(1 + \frac{1}{e}) \approx f(1) + \frac{1}{e} * f'(1) = \log_2(1) + \frac{1}{e} * \frac{1}{\ln(2)} \approx 0.5307$$

Therefore,

$$\log_2(\ln(e+1)) \approx 0.5307$$

I computed (with a calculator) $\log_2(\ln(e+1)) \approx 0.3931$

Problem 2 Find the slope of the tangent line at $(\frac{1}{2}, 2)$ to the curve:

$$arctan(xy) = \frac{\pi}{4}\frac{1}{xy}$$

Solution:
Doing *Implicit Differentiation*, we get

$$\frac{d}{dx}\left(arctan(xy) = \frac{\pi}{4}\frac{1}{xy}\right) \Leftrightarrow \frac{1}{1+(xy)^2}\left(y + x\frac{dy}{dx}\right) = \frac{\pi}{4}\left(\frac{-y-x\frac{dy}{dx}}{(xy)^2}\right)$$

Substituting $x = \frac{1}{2}, y = 2$,

$$\frac{dy}{dx} = \frac{1+\frac{\pi}{2}}{-\frac{1}{4}-\frac{\pi}{8}}$$

Problem 3 Study the function $f(x) = x - \ln(x) + x^{-1}$

(a) Compute the derivative and find extrema, if any.

Solution:
$f'(x) = 1 - \frac{1}{x} - x^{-2} = \frac{x^2-x-1}{x^2}$
Set $f'(x) = 0$, and solve for x.
Therefore $x = \frac{1\pm\sqrt{5}}{2}$ (by quadratic formula on the numerator of $f'(x)$).
But x must be positive, since $\ln(x)$ is defined for all $x > 0$.
Thus $x = \frac{1+\sqrt{5}}{2}$.

(b) Compute the second derivative and find inflection points.

Solution:
$f''(x) = \frac{1}{x^2} + \frac{2}{x^3} = \frac{x+2}{x^3}$
Set $f''(x) = 0$ and solve for x.
Thus $x = -2$, but f(-2) is not defined, so there are no inflection points.

Problem 4 Differentiate the functions (no need to simplify):

(a) $f(x) = \frac{x}{1+2^x}$

Solution:

$$f'(x) = \frac{1+2^x - x\ln(2)2^x}{(1+2^x)^2}$$

(b) $sin(x)^{cos(x)}$

Solution:
Let $y = sin(x)^{cos(x)}$, then
$\ln(y) = ln(sin(x)^{cos(x)}) = cos(x) * \ln(sin(x))$
Doing implicit differentiation:
$\frac{1}{y}\frac{dy}{dx} = \frac{cos^2(x)}{sin(x)} - sin(x)\ln(sin(x))$
Thus $\frac{dy}{dx} = \left(\frac{cos^2(x)}{sin(x)} - sin(x)\ln(sin(x))\right) sin(x)^{cos(x)}$

(c) $f(x) = tan(x)arcsin(x)$

Solution:

$$f'(x) = sec^2(x) * arcsin(x) + \frac{tan(x)}{\sqrt{1-x^2}}$$

Problem 5 Evaluate the following limits using any method:

(a)

$$\lim_{x\to\infty}\left(1+\frac{1}{2\sqrt{x}}\right)^{3\sqrt{x}}$$

Solution:

$$Recall: \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

Thus,

$$\lim_{x\to\infty}\left(1+\frac{1}{2\sqrt{x}}\right)^{3\sqrt{x}} = \lim_{x\to\infty}\left(1+\frac{1}{2\sqrt{x}}\right)^{3\sqrt{x}*\frac{2}{2}} = \lim_{x\to\infty}\left(1+\frac{1}{2\sqrt{x}}\right)^{2\sqrt{x}*\frac{3}{2}} = e^{\frac{3}{2}}$$

(b)

$$\lim_{x\to 0^+} \frac{e^x - 1}{\ln x}$$

Solution:

$$\lim_{x\to 0^+} lnx = -\infty$$

$$\lim_{x\to 0^+} e^x - 1 = 0.$$

$$Therefore \lim_{x\to 0^+} \frac{e^x - 1}{\ln x} = 0.$$

Problem 6 Consider the function $f(x) = 2^{3+\log_4 x^5} - 6$

(a) Use f' to show that f is a one-to-one function on $[1, \infty)$

Solution:
$f'(x) = \ln(2) * 2^{(3+\log_4 x^5)}(\frac{5x^4}{x^5 * \ln 4})$
Since x is in $[1, \infty)$ then $\log_4 x \geq 0$. Therefore $f'(x) > 0$. Thus f is one-to-one.

(b) Find the inverse function f^{-1} for f (you do not need to simplify).

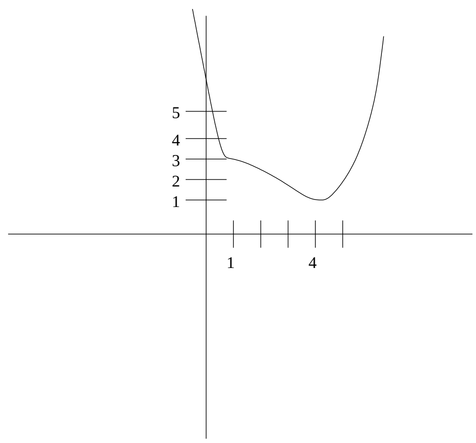
Solution:
Since f is one-to-one then f^{-1} exist, in this case we can swap $f(x)$ and x for our initial problem and solve for $f(x)$ which will be our f^{-1}.
$f^{-1}(x) = (4^{\log_2 x+6-3})^{\frac{1}{5}}$

Problem 7 Simplify as much as possible: $2^{\log_4 \frac{1}{3}+\log_4 48}$
Solution:
Using log rules:
$2^{\log_4 \frac{1}{3}+\log_4 48} = 2^{\log_4 \frac{48}{3}} = 2^{\log_4 16} = 2^2 = 4$

Problem 8 Sketch a graph of a differentiable function f that satisfies the following $f'(x) = 0$ only when $x = 1$ or 4; $f(1) = 3, f(4) = 1$; $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 4$ and $f''(x) > 0$ when $x < 1$

Solution:

This is a rough sketch, I couldn't draw it well.



Problem 9 The stiffness of a rectangular beam is proportional to the product of the width and the cube of the height of its cross section. What shape beam should be cut from a log in the form of a right circular cylinder of radius r in order to maximize its stiffness?

Solution:

We want to find the dimensions of the rectangular cross section of the beam that maximizes the stiffness. That ractangle should be circumscribed in a circle of radius r, where r is a constant throughout this problem.

Let h = height, and w = width of the ractangular cross section. By the Pythagorean Theorem,

$$\left(\frac{1}{2}h\right)^2 + \left(\frac{1}{2}w\right)^2 = r^2,$$

so $h^2 + w^2 = 4r^2$, and we get $w = \sqrt{4r^2 - h^2}$.

The stiffness is given by the function $y = wh^3$, so we have

$$y = (\sqrt{4r^2 - h^2})h^3.$$

To maximize y, first find the derivative of y:

$$\begin{aligned}
y' &= \frac{-2h}{2\sqrt{4r^2-h^2}}\, h^3 + \sqrt{4r^2-h^2}(3h^2) \\
&= -h^4 + (4r^2-h^2)(3h^2) \\
&= -h^4 + 12r^2h^2 - 3h^4 \\
&= 12r^2h^2 - 4h^4.
\end{aligned}$$

Then solving $y' = 0$ gives:

$$\begin{aligned}
12r^2h^2 - 4h^4 &= 0 \\
4h^2(3r^2 - h^2) &= 0 \\
h = 0 \text{ or } h = \sqrt{3}r
\end{aligned}$$

Since $h = 0$ gives zero stiffness, it cannot be the desired maximum stiffness. Hence, the height must be $\sqrt{3}r$ and the width is $w = \sqrt{4r^2 - h^2} = \sqrt{4r^2 - 3r^2} = r$.

Problem 10 A cannonball is fired from the ground straight up into the air at a velocity of 336 feet per second. Assuming constant acceleration due to gravity is 16 feet per second, write an equation describing the position of the cannonball after t seconds. Find the time that it hits the ground.

Solution:

$f(t) = -8t^2 + 336t$
$f(t) = 0 \Rightarrow t = 42$ seconds

Problem 11 Using the limit definition of the derivative, show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

Solution:

Recall:

$$f'(x) = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}$$

Note:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \tag{1}$$

Therefore we need to compute the left and right hand limits,

$$f'(0) = \lim_{\triangle x \to 0^+} \frac{\triangle x}{\triangle x} = 1$$

$$f'(0) = \lim_{\triangle x \to 0^-} \frac{-\triangle x}{\triangle x} = -1$$

Thus f is not differentiable at 0, since the limits from both sides do not equal.

Problem 12 Extra Credit: Evaluate

$$\lim_{x \to 0} \frac{e^{\frac{-1}{x^2}}}{x}$$

Solution:

$$\begin{aligned}
\lim_{x \to 0} \frac{e^{-\frac{1}{x^2}}}{x} &= \lim_{x \to 0} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} \\
&= \lim_{x \to 0} \frac{-\frac{1}{x^2}}{-\frac{2}{x^3}\ e^{\frac{1}{x^2}}} \quad \text{(using L'Hopital's rule)} \\
&= \frac{x}{2e^{\frac{1}{x^2}}} \\
&= 0
\end{aligned}$$

because the numerator goes to 0 and the denominator goes to ∞.