Intr.: Ernest Woei *August 30, 2006*

*Last name:*__________________ *First name:*__________________

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 7 pages, including this one.
2. **NO** calculators, books, notes or other written material allowed.
3. Express all numbers in exact arithmetic, i.e., no decimal approximations.
4. Read the statement below and sign your name.

   I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ______________________________

1. Find the indefinite integrals.

(a)

$$\int \csc(t^2+1)\cot(t^2+1)4t\ dt$$

Do integration by substitution

let $u = t^2+1$ $du = 2t\ dt$

$2\,du = 4t\ dt$

$$= \int \csc(u)\cot(u)\,2\,du = 2\csc(u) + C = 2\csc(t^2+1) + C$$

(b)

$$\int \frac{\sec u \tan u}{2\sec u - 1}\,du$$

Notice the derivative of the denominator is the numerator

let $v = 2\sec u - 1$ $dv = 2\sec u \tan u\ du$

$\frac{1}{2}\,dv = \sec u \tan u\ du$

$$= \int \frac{\frac{1}{2}}{v}\,dv = \frac{1}{2}\int \frac{1}{v}\,dv$$

$$= \frac{1}{2}\ln|v| + C = \frac{1}{2}\ln|2\sec u - 1| + C$$

(c)

$$\int e^{\cot x}\csc^2 x\ dx$$

Notice $\frac{d}{dx}\cot x = -\csc^2 x$

So we can do a substitution

let $u = \cot x$ $du = -\csc^2 x\ dx$

$-du = \csc^2 x\ dx$

$$= -\int e^u\,du = -e^u + C$$

$$= -e^{\cot x} + C$$

(d)

propertie of log.

$$\int \ln x^3\, dx = \int 3 \ln x \, dx$$

we can not do a substitution therefore we must do IBP

$$= 3\int \ln x \, dx$$

$u = \ln x$ $dv = dx$

$du = \frac{1}{x}\, dx$ $v = x$

$$= 3\left[x \ln x - \int x \frac{1}{x}\, dx \right] = 3\left[x \ln x - \int 1\, dx \right]$$

$$= 3\left[x \ln x - x \right] + C$$

(e)

$$\int x(x-1)^{4/3}\, dx$$

You can not simplify this integral & it does not look like we can integrate directly. Therefore we can either do a substitution or IBP. Try substitution first.

$$= \int (1+u)\, u^{4/3}\, du$$

$$= \int u^{4/3} + u^{7/3}\, du$$

$u = x - 1 \Longrightarrow 1 + u = x$

$du = dx$

$$= \frac{3u^{7/3}}{7} + \frac{3u^{10/2}}{10} + C = \frac{3(x-1)^{7/3}}{7} + \frac{3(x-1)^{10/3}}{10} + C$$

(f)

$$\int \frac{x}{e^x}\, dx$$

Use IBP

$u = x$ $dv = e^{-x}\, dx$

$du = dx$ $v = -e^{-x}$

$$= \int x e^{-x}\, dx = -xe^{-x} + \int e^{-x}\, dx$$

$$= -xe^{-x} - e^{-x} + C$$

2. Find the definite integrals.

(a)

$$\int_{-3}^{3} |2x+4| + 3x - 4\, dx = \int_{-3}^{3} |2x+4|\, dx + \underbrace{\int_{-3}^{3} 3x\, dx}_{= 0} - 4\underbrace{\int_{-3}^{3} dx}_{= 24}$$

since $3x$ is odd function and we are integrating on $[-3,3]$

So we need to just compute

$$\int_{-3}^{3} |2x+4|\, dx \qquad |2x+4| = \begin{cases} 2x+4 & 2x+4 \geq 0 \iff x \geq -2 \\ -(2x+4) & x \leq -2 \end{cases}$$

$$= \int_{-3}^{-2} -(2x+4)\, dx + \int_{-2}^{3} 2x+4\, dx = -x^2 - 4x \Big|_{-3}^{-2} + x^2 + 4x \Big|_{-2}^{3}$$

$$= -4+8-(-9+12) + (9+12) - (4-8)$$

$$= 1 + 25 = 26$$

therefore, $\int_{-3}^{3} |2x+4| + 3x - 4\, dx = 26 - 24 = 2$ //

(b)

$$\int_{-\pi}^{\pi} (x^4+1)\sin x\, dx$$

Notice $f(x) = (x^4+1)\sin x$ is an odd function

since $f(-x) = ((-x)^4+1)\sin(-x) = (x^4+1)(-\sin x) = -\sin x\,(x^4+1) = -f(x)$

thus integration over an interval $[-\pi, \pi]$ of an odd function results in having $\int_{-\pi}^{\pi} (x^4+1)\sin x\, dx = 0$ //

(c)

Here $x^2 \cos x$ is an even function

$$\int_{-\pi}^{\pi} x^2 \cos x \, dx = 2\int_0^{\pi} x^2 \cos x \, dx$$

$$\int x^2 \cos x \, dx \overset{IBP}{=} x^2 \sin x - \int \sin x \, (2x) \, dx \qquad \begin{array}{ll} u = x^2 & dv = \cos x \, dx \\ du = 2x \, dx & v = \sin x \end{array}$$

$$= x^2 \sin x - \left[-2x\cos x + \int 2\cos x \, dx \right] \qquad \begin{array}{ll} u = 2x & dv = \sin x \, dx \\ du = 2dx & v = -\cos x \end{array}$$

$$= x^2 \sin x + 2x\cos x \quad 2\sin x$$

so

$$2\int_0^{\pi} x^2 \cos x \, dx = 2\left[x^2 \sin x + 2x\cos x - 2\sin x \right]_0^{\pi}$$

$$= 2\left[2\pi \cos \pi - 0 \right] = -4\pi \;//$$

3. Find the area of the region bounded by the graphs: $y = -x^2 \ln x, y = 0, x = 1$, and $x = e$.

Note: $-x^2 \ln x$ is below the x-axis in the interval $[1, e]$

$$\text{Area} = \int_1^e 0 - (-x^2 \ln x) \, dx = \int_1^e x^2 \ln x \, dx$$

using IBP $\qquad \begin{array}{ll} u = \ln x & dv = x^2 dx \\ du = \frac{1}{x} dx & v = \frac{x^3}{3} \end{array}$

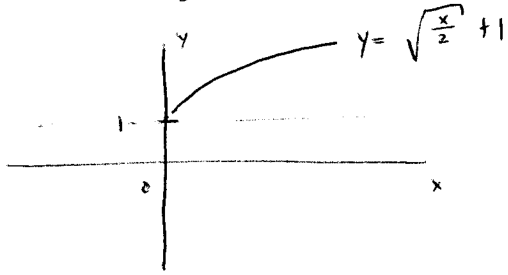
$$\int_1^e x^2 \ln x \, dx = \frac{x^3 \ln x}{3}\Big|_1^e - \int_1^e \frac{x^3}{3}\frac{1}{x} \, dx$$

$$= \frac{x^3 \ln x}{3}\Big|_1^e - \frac{x^3}{9}\Big|_1^e = \frac{e^3}{3} - \left[\frac{e^3}{9} - \frac{1}{9}\right] \;//$$

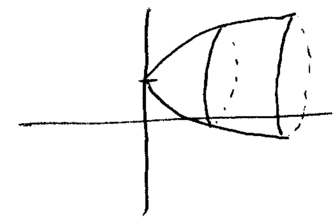
4. Find the volume of the solid formed by revolving the graph of

$$y = \sqrt{\frac{x}{2}} + 1, \quad 0 \le x \le 4$$

about the line $y = 1$.



revolve around $y=1$ $\longrightarrow$

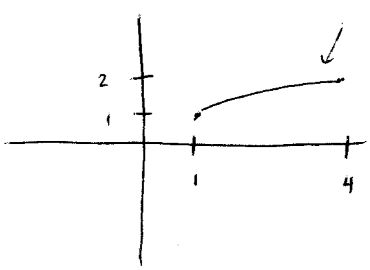


$$\text{Volume} = \pi \int_0^4 \left(\sqrt{\tfrac{x}{2}} + 1 - 1\right)^2 dx = \pi \int_0^4 \left(\sqrt{\tfrac{x}{2}}\right)^2 dx = \pi \int_0^4 \frac{x}{2}\, dx$$

$$= \frac{\pi}{2}\left[\frac{x^2}{2}\right]_0^4 = 4\pi \quad //$$

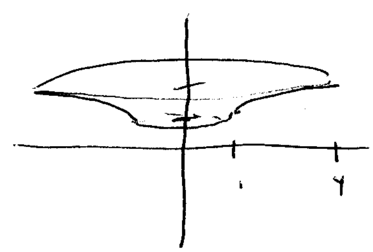
5. Find the volume of the solid formed by revolving the graph of $y = \sqrt{x}$ for $1 \le x \le 4$ about the y-axis.

$y = \sqrt{x}$



revolve around y-axis $\longrightarrow$

$y = \sqrt{x} \Rightarrow y^2 = x$

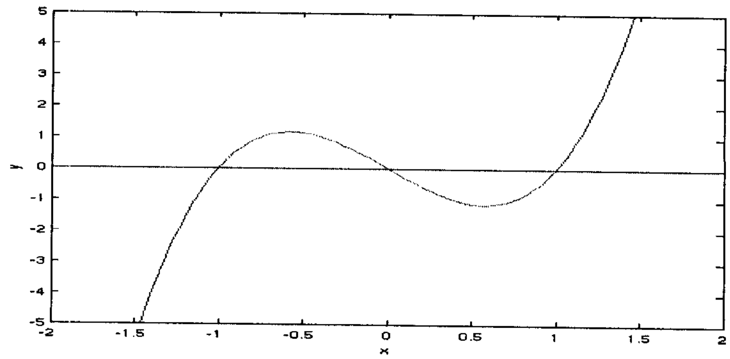


$$\text{Volume} = \pi \int_1^2 (y^2)^2\, dy = \pi \int_1^2 y^4\, dy$$

$$= \pi \left.\frac{y^5}{5}\right|_1^2 =$$

$$= \frac{\pi}{5}[31] = \frac{31\pi}{5} \quad //$$

6. Find the area of the region bounded by the graphs of $f(x) = 3(x^3 - x)$ and $g(x) = 0$. Hint: The graph of $f(x)$ and $g(x)$ is shown.



F & g intersect at $f(x) = g(x)$

$$3(x^3 - x) = 0 \Rightarrow 3x(x^2-1) = 0$$

$$3x(x-1)(x+1) = 0$$

$$\Rightarrow x = 0, 1, -1.$$

$$\text{Area} = \int_{-1}^{0} f(x) - g(x)\, dx + \int_{0}^{1} g(x) - f(x)\, dx$$

$$= 3\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + -3\left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1}$$

$$= 3\,\frac{1}{4} + -3\left(-\frac{1}{4}\right) = \frac{6}{4} = 3/2 \quad //$$

7. Find the average value of the function $f(x) = x\sqrt{4-x^2}$ over the interval $[0, 2]$. Find all x-values in the interval for which the function is equal to its average value.

$$u = 4 - x^2$$
$$du = -2x\, dx$$
$$-\frac{1}{2}\, du = x\, dx$$

$$\text{Average value} = \frac{1}{2-0} \int_0^2 x\sqrt{4-x^2}\, dx$$

$$= \left(\frac{1}{2}\right) 2\,\frac{(4-x^2)^{3/2}}{3}\left(-\frac{1}{2}\right)\Big|_0^2 = \frac{4^{3/2}}{6} = \frac{8}{6} = \frac{4}{3} \quad //$$

$$f(x) = \frac{4}{3} \Rightarrow \frac{4}{3} = x\sqrt{4-x^2} \Rightarrow \frac{16}{9} = x^2(4-x^2) \Rightarrow \frac{16}{9} = 4x^2 - x^4$$

$$x^4 - 4x^2 + \frac{16}{9} = 0 \Rightarrow x = \sqrt{2\left(1 \pm \frac{\sqrt{5}}{3}\right)}$$