Instr.: Ernest Woei *July 22, 2005*

*Last name:*____________________ *First name:*____________________

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 6 pages, including this one.
2. **NO** calculators, books, notes or other written material allowed.
3. Express all numbers in exact arithmetic, i.e., no decimal approximations.
4. Read the statement below and sign your name.

   I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ______________________________________

"Anyone who has never made a mistake
has never tried anything new." –Albert Einstein.

GOOD LUCK!!!

1. (*4 pts*) Fill each of the underlined blank spaces with the correct number.

   (a) (*1 pt*) Evaluate the determinant:

$$\begin{vmatrix} 2 & 1 & 15 & 5 & 1 \\ 4 & 2 & 12 & 4 & 2 \\ 6 & 3 & 9 & 3 & 3 \\ 8 & 4 & 6 & 2 & 4 \\ 10 & 5 & 3 & 1 & 5 \end{vmatrix} = \underline{0}$$

2 columns are the same $\Longrightarrow$ determinant is 0

   (b) (*1 pt*) $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 5 & 5 & 3 \\ 1 & 4 & 5 & -2 \\ 2 & 5 & 6 & 2 \end{bmatrix}$ has LU decomposition, $LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

$$det(A) = \underline{1}$$

$\det(A) = \det(LU) = \det(L)\det(U) = 1 \cdot 1 = 1$

   (c) (*2 pts*) Evaluate the determinant:

$$\begin{vmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{vmatrix} = \underline{44}$$

Cofactor expansion along 2nd row

$2(-1)^{2+3} \det\left(\begin{bmatrix} 0 & 2 & 0 \\ 4 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}\right) = 44.$

cofactor expansion along 1st row

$2(-1)^{1+2} \det\left(\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\right) = -22$

2. (*9 pts*) For each of the following statements/questions, circle the correct answers.

   (a)
$$\begin{array}{rcrcrcrcrcl} 2x_1 & + & x_2 & + & 15x_3 & + & 5x_4 & + & x_5 & = & 0 \\ 4x_1 & + & 2x_2 & + & 12x_3 & + & 4x_4 & + & 2x_5 & = & 0 \\ 6x_1 & + & 3x_2 & + & 9x_3 & + & 3x_4 & + & 3x_5 & = & 0 \\ 8x_1 & + & 4x_2 & + & 6x_3 & + & 2x_4 & + & 4x_5 & = & 0 \\ 10x_1 & + & 5x_2 & + & 3x_3 & + & x_4 & + & 5x_5 & = & 0 \end{array}$$
has a

Nontrivial Solution OR *Only Trivial Solution*?

   (b) Let $v_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Are these vectors orthogonal? *YES* *NO*

   (c) Let $v = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. Is v a unit vector in $\mathbb{R}^2$? *YES* *NO*

   (d) If V is a vector space that has a *nonzero vector*, how many vectors are in V?

1 *OR* 2 *OR* 3 *OR* *More than 4*?

(e) Given these sets of vectors, determine whether they are linearly independent or linearly dependent.

i. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ is *Linearly Independent* *Linearly Dependent*

ii. $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 3 \\ -3 \end{bmatrix} \right\}$ is *Linearly Independent* *Linearly Dependent*

iii. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is *Linearly Independent* *Linearly Dependent*

iv. $\left\{ \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is *Linearly Independent* *Linearly Dependent*

v. $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ is *Linearly Independent* *Linearly Dependent*

3. (*5 pts*) Evaluate the determinant:

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = (b-a)(\underline{c} - a)(\underline{d} - a)(\underline{c} - b)(\underline{d} - b)(\underline{d} - c)$$

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix} = (b-a)(c-a)(d-a) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ab+a^2 \\ 0 & 1 & c+a & c^2+ac+a^2 \\ 0 & 1 & d+a & d^2+ad+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(d-a) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ab+a^2 \\ 0 & 0 & c-b & (c-b)(a+b+c) \\ 0 & 0 & d-b & (d-b)(a+b+d) \end{vmatrix} = (b-a)(c-a)(d-a)(c-b)(d-b) \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ab+a^2 \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 1 & a+b+d \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ab+a^2 \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 0 & d-c \end{vmatrix}$$

$$= (b-a)(c-a)(d-a)(c-b)(d-b)(d-c)\underbrace{\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & b^2+ab+a^2 \\ 0 & 0 & 1 & a+b+c \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{=1}$$

4. (*4 pts*) Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is a *linear transformation* with

$$L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\2\end{bmatrix}, \quad L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\3\end{bmatrix}$$

(a) (*2 pts*) $L\left(\begin{bmatrix}2\\3\\1\end{bmatrix}\right) = \begin{bmatrix}1\\4\\7\end{bmatrix}$

$$2\begin{bmatrix}1\\0\\-1\end{bmatrix} + 3\begin{bmatrix}0\\1\\2\end{bmatrix} + 1\begin{bmatrix}-1\\1\\3\end{bmatrix}$$

(b) (*1 pt*) Find the standard matrix representing L, i.e. $L(x) = Ax$

$$A = \begin{bmatrix}1 & 0 & -1\\0 & 1 & 1\\-1 & 2 & 3\end{bmatrix}$$

(c) (*1 pt*) Is $\begin{bmatrix}2\\1\\1\end{bmatrix}$ in the range of L? *YES* *NO*

If so, enter the vector in $\mathbb{R}^3$ such that $L(x) = \begin{bmatrix}2\\1\\1\end{bmatrix}$, $x = \begin{bmatrix} \\ \\ \end{bmatrix}$

$$\left[\begin{array}{ccc|c}1 & 0 & -1 & 2\\0 & 1 & 1 & 1\\-1 & 2 & 3 & 1\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & -1 & 2\\0 & 1 & 1 & 1\\0 & 2 & 2 & 3\end{array}\right] \to \left[\begin{array}{ccc|c}1 & 0 & -1 & 2\\0 & 1 & 1 & 1\\0 & 0 & 0 & 1\end{array}\right]$$

$\Rightarrow \quad 0x_1 + 0x_2 + 0x_3 = 1 \quad \times \quad \Rightarrow$ no x.

5. (*4 pts*) Let $S = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -5 \end{bmatrix}, \quad v_4 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 8 \end{bmatrix},$$

define $W = span\ S$.

Find a basis for the subspace W.

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ -2 & 1 & 0 & 1 \\ 0 & -1 & 2 & 3 \\ -1 & 3 & -5 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & 3 \\ 0 & 3 & -6 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$\{v_1, v_2, v_4\}$ is a basis for W $\Leftarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

What is the dimension of this subspace? $dim\ W =$ 3

6. (*3 pts*) Let A be an $m \times n$ matrix. Show that the null space of A is a subspace of $\mathbb{R}^n$.

Null space of A = $N(A)$ = $\{$ sol'ns to the homogeneous system $\}$, $0_{\mathbb{R}^n}$ is in $N(A)$

(α) x, y in $N(A)$ so $A(x+y) = Ax + Ay = 0_{\mathbb{R}^m} + 0_{\mathbb{R}^m} = 0_{\mathbb{R}^m}$

$\Rightarrow$ $x+y$ in $N(A)$

(β) c in $\mathbb{R}$, x in $N(A)$ so $A(cx) = c\,Ax = c\,0_{\mathbb{R}^m} = 0_{\mathbb{R}^m}$

$\Rightarrow$ cx in $N(A)$

thus $N(A)$ is a subspace of $\mathbb{R}^n$.

7. *(1 pt)* Let $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & -1 & 2 \\ 0 & -1 & 6 \end{bmatrix}$. Find a basis for the null space of A.

$$\left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 2 & -1 & 2 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 4 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & -1 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_3 = 0$

$x_2 - 6x_3 = 0$

x_3 free

$x_3 = s$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2s \\ 6s \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}$$

is a sol'n to the homogenous system.

then $\left\{ \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix} \right\}$ is a basis for the null spac of A.

Page	2 *(8)*	3 *(10)*	4 *(4)*	5 *(7)*	6 *(1)*	Total *(30)*
Scores						