Intr.: Ernest Woei *July 8, 2005*

*Last name:*__________________ *First name:*__________________

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 5 pages, including this one.
2. **NO** calculators, books, notes or other written material allowed.
3. Express all numbers in exact arithmetic, i.e., no decimal approximations.
4. Read the statement below and sign your name.

I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ________________________________

*"You can profit from your mistakes,
but that does not mean the more mistakes, the more profit." – Anonymous*

GOOD LUCK!!!

1. (*6 pts*) For each of the following statements/questions. Circle the correct answers.

   (a) Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, is A a diagonal matrix? *YES* *NO*

   (b) Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 3 \\ 1 & 2 & 1 \end{bmatrix}$, is A a symmetric matrix? *YES* *NO*

   (c) Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, is A in reduced row echelon form? *YES* *NO*

   (d) Let A be a 3×3 matrix. Suppose that $x = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ is a solution to the homogeneous system $Ax = 0$. Is A nonsingular or singular? *Nonsingular* *Singular*

   (e) Is 132 a permutation of the set $S = \{1, 2, 3, 4\}$? *YES* *NO*

   (f) Let A, B be $n \times n$ matrices. Suppose A, B, $A + B$ are nonsingular. Is it necessarily true that $(A + B)^{-1} = A^{-1} + B^{-1}$? *YES* *NO*

2. (*7 pts*) Fill each of the underlined blank spaces with the appropriate number or symbol.

   (a) (*2 pts*) Solve: $\begin{cases} x + 2y = 0 \\ 2x + 3y = 1 \end{cases}$ Answer: $\begin{cases} x = \underline{2} \\ y = \underline{-1} \end{cases}$

   (b) (*3 pts*) Let $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$, $x = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$, $Ax = \begin{bmatrix} 15 \\ 27 \\ 33 \end{bmatrix} = b$. Write b as a linear combination of columns of A.

   Answer: $\left\{ \underline{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \underline{3} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + \underline{1} \begin{bmatrix} 1 \\ 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 15 \\ 27 \\ 33 \end{bmatrix} \right.$

   (c) Let P & A be $n \times n$ matrices. P nonsingular. Define $C = PAP^{-1}$. Give a simple expression for $C^{10} = \underline{P A^{10} P^{-1}}$

   (d) Given the permutation 35142 of the set $S = \{1, 2, 3, 4, 5\}$. How many inversion(s) does the permutation have? $\underline{6}$ inversion(s).

3. (*3 pts*) Evaluate:

(a) $\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}^{-1} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

4. (*3 pts*) Let B be the 4×4 matrix whose LU decomposition is given by

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the solution to the linear system, $Bx = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Work (*2 pts*) : $B = LU$ $\quad Bx = LUx = b$

Let $y = Ux \Rightarrow Ly = b \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 1 \end{array}\right]$

$y_1 = 1$

$\Rightarrow y_2 = 1 - 2y_1 = -1$

$y_3 = 1 - y_1 - 2y_1 = 2$

$y_4 = 1 - 2y_1 - y_2 - y_3 = -2$

$\Rightarrow Ux = y \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right]$

$x_4 = -2$

$\Rightarrow x_3 = 2 + 2x_4 = -2$

$x_2 = -1 + x_4 - x_3 = -1$

$x_1 = 1 - 2x_4 - 2x_3 - 2x_2$

$= 11$

(b) Answer (*1 pt*): $x = \begin{bmatrix} 11 \\ -1 \\ -2 \\ -2 \end{bmatrix}$

5. (*5 pts*) Find the solution to the system of linear equations $Ax = b$, where

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 0 & -1 & 0 & 1 & -1 \\ 2 & 2 & 2 & 8 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix}. \left\{ \begin{array}{l} x_1 = \underline{3+3r} \\ x_2 = \underline{-2r-1} \\ x_3 = \underline{3+3r} \\ x_4 = \underline{-r-1} \\ x_5 = \underline{r} \end{array} \right\}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 0 & 2 \\ 0 & -1 & 0 & 1 & -1 & 0 \\ 2 & 2 & 2 & 8 & 0 & 2 \\ -1 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\substack{-2r_1+r_3 \to r_3 \\ -r_2 \to r_2}]{r_1+r_4 \to r_4} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & -2 & 0 & 4 & 0 & -2 \\ 0 & 2 & 2 & 2 & 0 & 0 \end{array}\right] \xrightarrow[-2r_2+r_4 \to r_4]{2r_2+r_3 \to r_3} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 2 & 4 & -2 & 2 \end{array}\right]$$

$$\xrightarrow[\substack{-2r_4+r_3 \to r_3 \\ r_4+r_2 \to r_2}]{-2r_4+r_1 \to r_1} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-r_3+r_1 \to r_1} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right] \xrightarrow{-2r_2+r_1 \to r_1} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & 3 & 3 \\ 0 & 0 & 0 & 1 & 1 & -1 \end{array}\right]$$

$x_1 = 3 + 3x_5 = 3 + 3r$

$x_2 = -1 - 2x_5 = -2r - 1$

$x_3 = 3 + 3x_5 = 3 + 3r$

$x_4 = -1 - x_5 = -1 - r$

$x_5 = r$

6. (*3 pts*) If R is obtained from A by a sequence of row operations:

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 3 \\ 2 & -2 & -6 \end{bmatrix} \xrightarrow{r_1+r_2\to r_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 2 & -2 & -6 \end{bmatrix} \xrightarrow{-2r_1+r_3\to r_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{2r_2+r_3\to r_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

where A is the first matrix in the sequence and R is the last matrix in the sequence. Write the product of elementary matrices that result in R, i.e.

$$E_n E_{n-1} \dots E_2 E_1 A = R,$$

where E_i are elementary matrices.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_3 E_2 E_1 A = R$$

7. (*3 pts*) Let A be an $n \times n$ matrix, and let x & y be vectors in $\mathbb{R}^n$. Show that if A satisfies the condition $AA^T = A^T A = I_n$, then

$$x \cdot y = (Ax) \cdot (Ay)$$

$$Ax \cdot Ay = (Ax)^T Ay = x^T A^T A y = x^T I_n y = x^T y = x \cdot y$$

Page	2 (*13*)	3 (*6*)	4 (*5*)	5 (*6*)	Total (*30*)
Scores					