Instr.: Ernest Woei *August 5, 2005*

*Last name:*____________________ *First name:*____________________

PLEASE READ THIS BEFORE YOU DO ANYTHING ELSE!

1. Make sure that your exam contains 8 pages, including this one.
2. **NO** calculators, books, notes or other written material allowed.
3. Express all numbers in exact arithmetic, i.e., no decimal approximations.
4. Read the statement below and sign your name.

I affirm that I neither will give nor receive unauthorized assistance on this examination. All the work that appears on the following pages is entirely my own.

Signature: ________Key_____________________

GOOD LUCK!!!

1. (*5 pts*) Consider the vector subspace W of $\mathbb{R}^4$ with basis

$$\left\{ \underset{= v_1}{\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}}, \underset{= v_2}{\begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix}}, \underset{= v_3}{\begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix}} \right\}.$$

Use the *Gram-Schmidt* orthogonalization procedure to find an orthonormal basis for W.

Work (*2 pts*):

$$u_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = v_1 \qquad u_2 = v_2 - \frac{v_2 \cdot w_1}{w_1 \cdot w_1} w_1 = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} - \frac{6}{1}\,\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$w_1 = \frac{1}{8}\begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$w_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad u_3 = v_3 - \frac{v_3 \cdot w_1}{w_1 \cdot w_1} w_1 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2$$

$$= \begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix} - \underbrace{\frac{8/2}{1}\,\frac{1}{2}}_{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \underbrace{\frac{8/2}{1}\,\frac{1}{2}}_{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

$$w_3 = \qquad = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Answer (*3 pts*):

$$\mathrm{w}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathrm{w}_2 = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathrm{w}_3 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

2. (*3 pts*) Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$.

Work (*2 pts*):

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & -2 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{vmatrix}$$

$$= (-1)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & -1 \end{vmatrix}$$

Answer (*1 pt*): det A = $\underline{-1}$.

3. (*2 pts*) Let $L : V \to \mathbb{R}^5$ be a linear transformation. If L is onto and $dim(ker\ L) = 2$, what is $dim\ V$? Justify your answer.

L onto $\rightarrow$ range $L = \mathbb{R}^5$

dim ker L + dim range L = dim V

$7 = 2 + 5 =$ dim V

4. (*4 pts*) Find the characteristic polynomial and eigenvalues of

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Work (*1 pt*):

$$\begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{vmatrix} = \begin{vmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ 0 & -\lambda & \lambda \end{vmatrix} = \begin{vmatrix} \lambda & -\lambda & 0 \\ -1 & \lambda-1 & -1 \\ 0 & -\lambda & \lambda \end{vmatrix}$$

$$-\lambda(-1)^{3+2}\begin{vmatrix} \lambda & 0 \\ -1 & -1 \end{vmatrix} + \lambda(-1)^{3+3}\begin{vmatrix} \lambda & -\lambda \\ -1 & \lambda-1 \end{vmatrix}$$

$$\lambda(-\lambda) + \lambda(\lambda(\lambda-1)-\lambda) = \lambda(-\lambda + \lambda(\lambda-1) - \lambda) = \lambda^2(-1+\lambda-1-1)$$
$$= \lambda^2(\lambda-3)$$

Answer (*3 pts*).
The characteristic polynomial = $\lambda^2(\lambda-3)$.

The eigenvalues: $\lambda =$ $0, \ 3$.

5. (*3 pts*) Let W be the set of $n \times n$ symmetric matrices.

(a) (*2 pts*) Show W is a subspace of the vector space M_{nn} ($n \times n$ matrices).

$n \times n$ symmetric matrices are a subset of $n \times n$ matrices

0_{nn} is a symmetric matrix so W is nonempty

① Let A, B be $n \times n$ symmetric matrices

then $A+B = A^T + B^T = (A+B)^T$ so $A+B$ is symmetric

$\Rightarrow$ $A+B$ in W

② Let c be a real number so

$cA = cA^T = (cA)^T$ so cA is a symmetric matrix $\Rightarrow$ cA in W

Since ① & ② are true then W is a subspace of M_{nn}

(b) (*1 pt*) $\dim W =$ $\frac{n(n+1)}{2}$.

$$\begin{bmatrix} & 1 & \\ 1 & & \end{bmatrix}$$ basis type elements

$$\begin{bmatrix} 0 & & \\ & \ddots & \\ n & & 0 \end{bmatrix}$$

$n + \underbrace{1 + \cdots + n-1}$

$n + \frac{(n-1)(n)}{2}$

$\frac{2n + n(n-1)}{2} = \frac{n(n+1)}{2}$

6. (*7 pts*) Consider the following 4×4 matrix A and linear transformation $L : \mathbb{R}^4 \to \mathbb{R}^4$:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(a) What is the rank of the matrix A? (*1 pt*) 1

(b) Find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ for the kernel of L, and a basis $\{\mathbf{u}_4\}$ for the range of L. (Work counts 1 point, and the answer counts 4 points.)

A is row equivalent to $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 + x_3 + x_4 = 0$ $\qquad$ x_2, x_3, x_4 are free

$x_1 = -x_2 - x_3 - x_4$

$= -r - s - t$

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$L(\vec{x}) = \begin{bmatrix} x_1 + x_2 + x_3 + x_4 \\ \vdots \\ \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \cdots + x_4 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\Downarrow$

$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is a basis for the range of L.

Answer: $\mathbf{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$; $\mathbf{u}_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Is $\mathbf{u}_4$ an eigenvector of A? (1 point, circle the right answer) *YES* *NO*

7. (*3 pts*) Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \textit{and} \quad T = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

be two bases for $\mathbb{R}^3$. Find the transition matrix $P_{S \leftarrow T}$ from the T-bases to the S-bases. Work (*1 pt*):

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\downarrow$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Answer (*2 pts*): $P_{S \leftarrow T} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

8. (*5 pts*) Let A be a real symmetric matrix of size $n \times n$. Assume that A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be corresponding eigenvectors such that $A\mathbf{v}_j = \lambda_j \mathbf{v}_j$.

   (a) (*2 pts*) Show that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$.

$$\lambda_i v_i \cdot v_j = A v_i \cdot v_j = v_i^T A^T v_j = v_i^T A v_j = v_i \cdot \lambda_j v_j$$

$$= \lambda_j (v_i \cdot v_j)$$

$\Rightarrow \quad (\lambda_i - \lambda_j)(v_i \cdot v_j) = 0$ since λ_i, λ_j are distinct then $v_i \cdot v_j$ must be 0.

$\boxed{2}$

   (b) (*2 pts*) Define $\mathbf{w}_j = \frac{\mathbf{v}_j}{\|\mathbf{v}_j\|}$ for every $j = 1, \ldots, n$. Show that $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ is linearly independent.

$$\sum_{i=1}^{n} c_i w_i = 0_{\mathbb{R}^n}$$

$$w_j \cdot \sum_{i=1}^{n} c_i w_i = w_j \cdot 0_{\mathbb{R}^n} = 0$$

$$\Downarrow$$

$$c_j (w_j \cdot w_j) = 0$$

$$c_j \cdot 1 = 0$$

so $c_j = 0$

for each j

thus $\{w_1, \ldots, w_n\}$ is linearly independent.

   (c) (*1 pt*) Define a matrix $P = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \cdots \quad \mathbf{w}_n]$. Find $P^{-1}AP$.

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}.$$

9. (5 pts) For each of the following statements, determine if it is true or false, and circle the correct answer.

(a) Let A be an $n \times n$ matrix and consider the linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$ defined by the matrix multiplication $L(x) = Ax$. If $det(A) \neq 0$, then L is one-to-one and onto.

TRUE *FALSE*

(b) Any collection of $n+1$ vectors in $\mathbb{R}^n$ is linearly dependent.

TRUE *FALSE*

(c) Let $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and $\mathbf{v}$ a vector in $\mathbb{R}^n$. Then the coordinate vector of $\mathbf{v}$ with respect to this basis is $(\mathbf{w}_1 \cdot \mathbf{v}, \mathbf{w}_2 \cdot \mathbf{v}, \ldots, \mathbf{w}_n \cdot \mathbf{v})^T$.

TRUE *FALSE*

(d) An arbitrary straight line or an arbitrary plane is an example of a vector subspace of $\mathbb{R}^3$.

TRUE *FALSE*

(e) Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be a basis for $\mathbb{R}^n$, where each vector is represented by a column vector. Define an $n \times n$ matrix $A = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \cdots \quad \mathbf{u}_n]$. Then $det(A) = 0$.

TRUE *FALSE*

10. (2 pts) Let $L : P_1 \to P_1$ be a linear transformation defined by

$$L(t+1) = t-1, \quad L(t-1) = 2t+1.$$

Find the matrix of L with respect to the basis $S = \{t+1, t-1\}$ for P_1.

$$\begin{bmatrix} [L(t+1)]_S & [L(t-1)]_S \end{bmatrix}$$

$$= \begin{bmatrix} [t-1]_S & [2t+1]_S \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{3}{2} \\ 1 & \frac{1}{2} \end{bmatrix}$$

is the matrix of L w.r.t to the S-basis

$2t+1 = c_1(t+1) + c_2(t-1)$

$= (c_1+c_2)t + c_1 - c_2$

$\Rightarrow$

$c_1 + c_2 = 2$ $\Rightarrow$ $2c_1 = 3$

$c_1 - c_2 = 1$

$c_1 = \frac{3}{2}$

$c_2 = \frac{1}{2}$

11. (1 pt) Find the least squares line for the given data points: $(-2,1),(-1,2),(1,3),(3,2)$.

$$\begin{bmatrix} \sum_{i=1}^{4} x_i^2 & \sum_{i=1}^{4} x_i \\ \sum_{i=1}^{4} x_i & \sum_{i=1}^{4} 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} x_i y_i \\ \sum_{i=1}^{4} y_i \end{bmatrix} \qquad ax + b = y.$$

$$\left[\begin{array}{cc|c} 15 & 1 & 5 \\ 1 & 4 & 8 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & -59 & -115 \\ 1 & 4 & 8 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 0 & 1 & \frac{115}{59} \\ 1 & 0 & -4 \cdot \frac{115}{59} + 8 \end{array}\right]$$

$$a = -4 \, \frac{115}{59} + 8 \qquad b = \frac{115}{59}$$

Page	2 (*5*)	3 (*5*)	4 (*7*)	5 (*10*)	6 (*5*)	7 (*7*)	8 (*1*)	Total (*40*)
Scores								